

Math 150 Calculus Theorems and Formulas

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. Scalar multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow L} f(x) = f(L)$, then

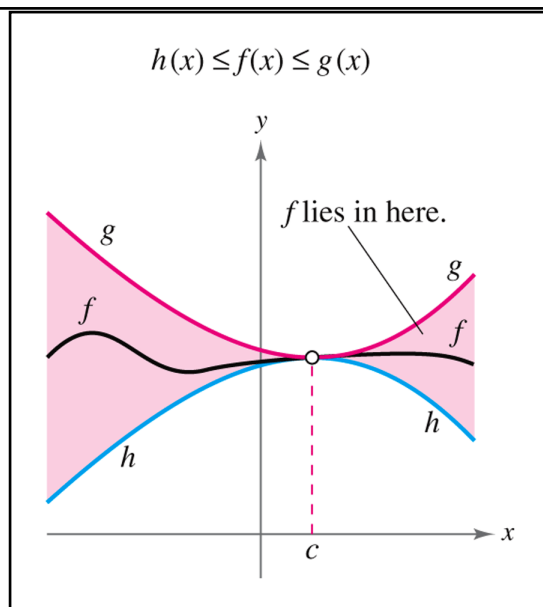
$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

THEOREM 1.8 The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

then $\lim_{x \rightarrow c} f(x)$ exists and is equal to L .



Summary of Differentiation Rules

General Differentiation Rules

Let f , g , and u be differentiable functions of x .

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Product Rule:

$$\frac{d}{dx}[fg] = fg' + gf'$$

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

Derivatives of Algebraic Functions

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u) u'$$

Sum or Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \quad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \quad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1} u'$$

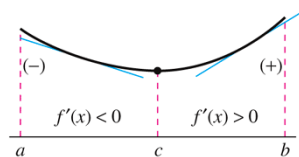
Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to* x .
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx .

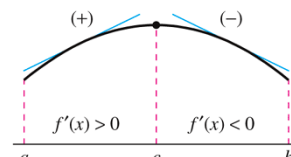
THEOREM 3.6 The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c . If f is differentiable on the interval, except possibly at c , then $f(c)$ can be classified as follows.

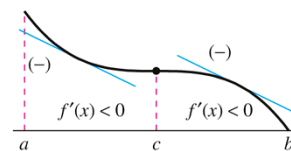
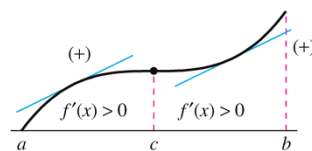
1. If $f'(x)$ changes from negative to positive at c , then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c , then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c , then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum



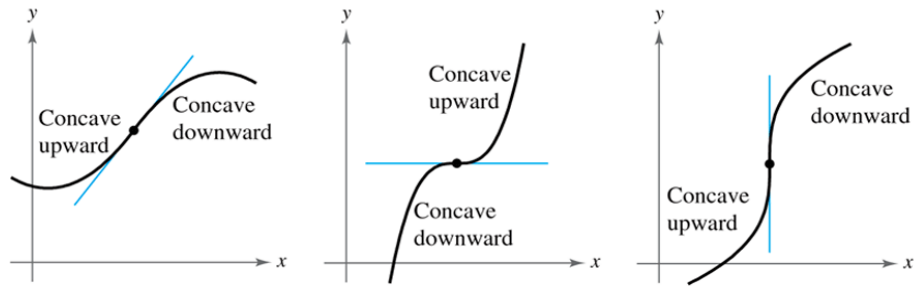
Relative maximum



Neither relative minimum nor relative maximum

Definition of Point of Inflection

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.



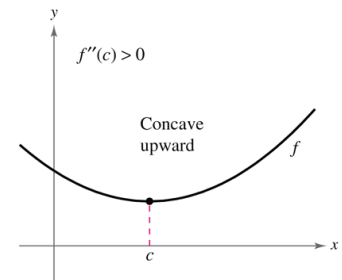
The concavity of f changes at a point of inflection. Note that a graph crosses its tangent line at a point of inflection.

THEOREM 3.9 Second Derivative Test

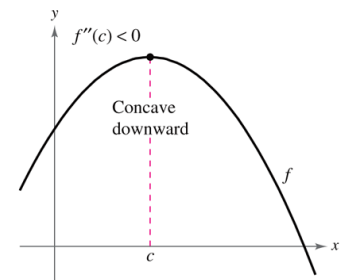
Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c .

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum at c , a relative minimum at $(c, f(c))$, or neither. In such cases, you can use the First Derivative



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x . The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x) dx.$$

Differential Formulas

Let u and v be differentiable functions of x .

Constant multiple: $d[cu] = c du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u dv + v du$

Quotient: $d\left[\frac{u}{v}\right] = \frac{v du - u dv}{v^2}$

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x) dx.$$

(See Figure 4.22.)

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions of kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$

2. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a , then, for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x).$$

THEOREM 4.12 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then

$$\int f(g(x))g'(x) dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x) dx$ and

$$\int f(u) du = F(u) + C.$$

Guidelines for Making a Change of Variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x) dx$.
3. Rewrite the integral in terms of the variable u .
4. Find the resulting integral in terms of u .
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x .
6. Check your answer by differentiating.

THEOREM 4.14 Change of Variables for Definite Integrals

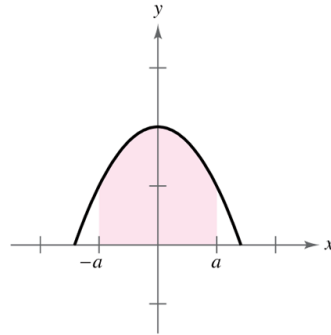
If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

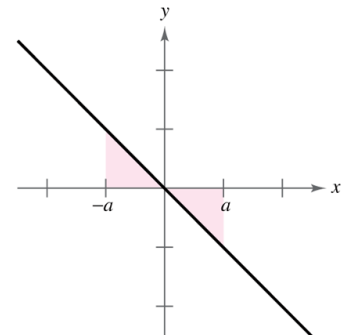
THEOREM 4.15 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
2. If f is an *odd* function, then $\int_{-a}^a f(x) dx = 0$.



Even function



Odd function

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x .

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x .

1. $\int \frac{1}{x} dx = \ln|x| + C$
2. $\int \frac{1}{u} du = \ln|u| + C$

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

THEOREM 5.11 Derivative of the Natural Exponential Function

Let u be a differentiable function of x .

1. $\frac{d}{dx}[e^x] = e^x$
2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x .

1. $\int e^x dx = e^x + C$
2. $\int e^u du = e^u + C$

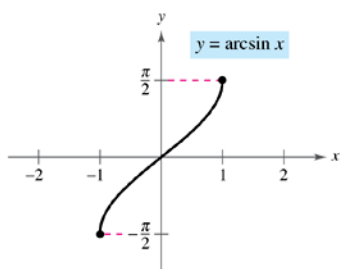
THEOREM 5.13 Derivatives for Bases Other Than e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x .

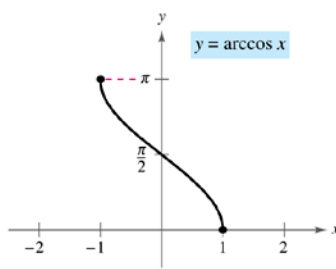
1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

Definitions of Inverse Trigonometric Functions

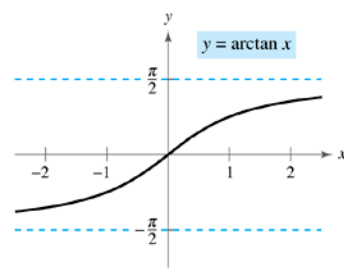
<i>Function</i>	<i>Domain</i>	<i>Range</i>
$y = \arcsin x$ iff $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot } x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec } x$ iff $\sec y = x$	$ x \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
$y = \text{arccsc } x$ iff $\csc y = x$	$ x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



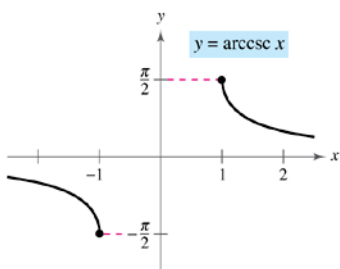
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



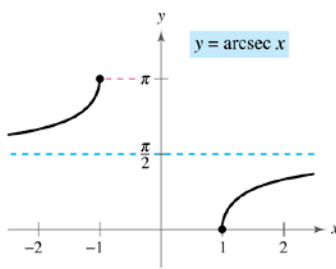
Domain: $[-1, 1]$
Range: $[0, \pi]$



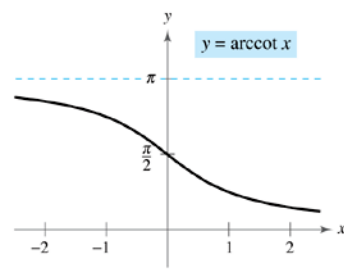
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Basic Differentiation Rules for Elementary Functions

$$1. \frac{d}{dx}[cu] = cu'$$

$$2. \frac{d}{dx}[u \pm v] = u' \pm v'$$

$$3. \frac{d}{dx}[uv] = uv' + vu'$$

$$4. \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$5. \frac{d}{dx}[c] = 0$$

$$6. \frac{d}{dx}[u^n] = nu^{n-1}u'$$

$$7. \frac{d}{dx}[x] = 1$$

$$8. \frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$$

$$9. \frac{d}{dx}[\ln u] = \frac{u'}{u}$$

$$10. \frac{d}{dx}[e^u] = e^u u'$$

$$11. \frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$$

$$12. \frac{d}{dx}[a^u] = (\ln a)a^u u'$$

$$13. \frac{d}{dx}[\sin u] = (\cos u)u'$$

$$14. \frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$15. \frac{d}{dx}[\tan u] = (\sec^2 u)u'$$

$$16. \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$17. \frac{d}{dx}[\sec u] = (\sec u \tan u)u'$$

$$18. \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

$$19. \frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$20. \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$21. \frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$22. \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$23. \frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$24. \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Basic Integration Rules ($a > 0$)

- $\int kf(u) du = k \int f(u) du$
- $\int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$
- $\int du = u + C$
- $\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
- $\int \frac{du}{u} = \ln|u| + C$
- $\int e^u du = e^u + C$
- $\int a^u du = \left(\frac{1}{\ln a}\right)a^u + C$
- $\int \sin u du = -\cos u + C$
- $\int \cos u du = \sin u + C$
- $\int \tan u du = -\ln|\cos u| + C$
- $\int \cot u du = \ln|\sin u| + C$
- $\int \sec u du = \ln|\sec u + \tan u| + C$
- $\int \csc u du = -\ln|\csc u + \cot u| + C$
- $\int \sec^2 u du = \tan u + C$
- $\int \csc^2 u du = -\cot u + C$
- $\int \sec u \tan u du = \sec u + C$
- $\int \csc u \cot u du = -\csc u + C$
- $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
- $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

Procedures for Fitting Integrands to Basic Rules

Technique

Expand (numerator).

Separate numerator.

Complete the square.

Divide improper rational function.

Add and subtract terms in numerator.

Use trigonometric identities.

Multiply and divide by Pythagorean conjugate.

Example

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$$

$$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$$

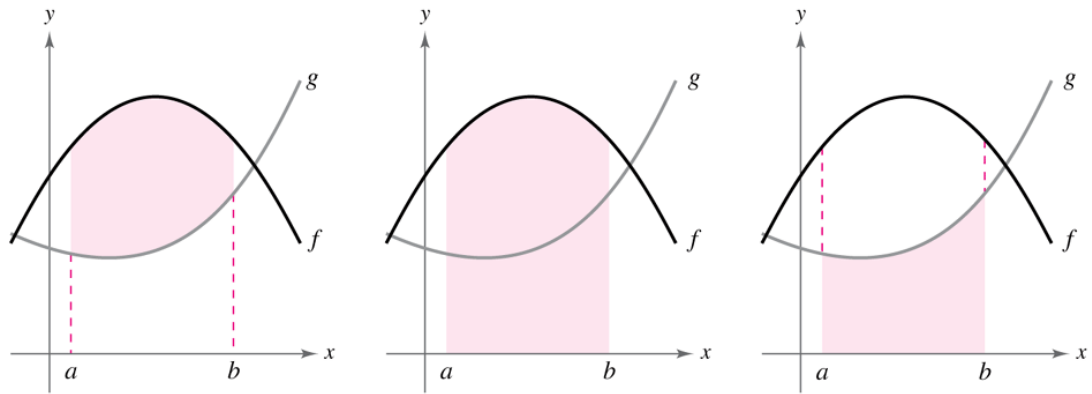
$$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$$

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

$$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$$

$$\cot^2 x = \csc^2 x - 1$$

$$\begin{aligned} \frac{1}{1+\sin x} &= \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x} \\ &= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x} \end{aligned}$$



Area of region
between f and g

=

Area of region
under f

-

Area of region
under g

$$\int_a^b [f(x) - g(x)] dx$$

=

$$\int_a^b f(x) dx$$

-

$$\int_a^b g(x) dx$$

The Disk Method

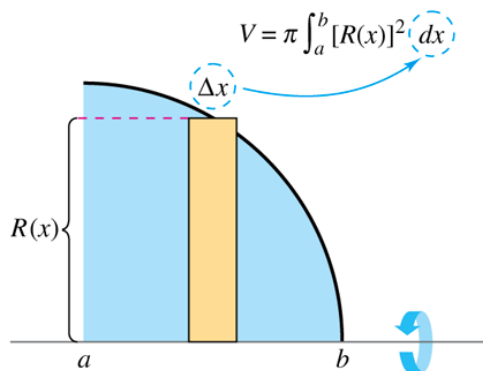
To find the volume of a solid of revolution with the **disk method**, use one of the following, as shown in Figure 7.15.

Horizontal Axis of Revolution

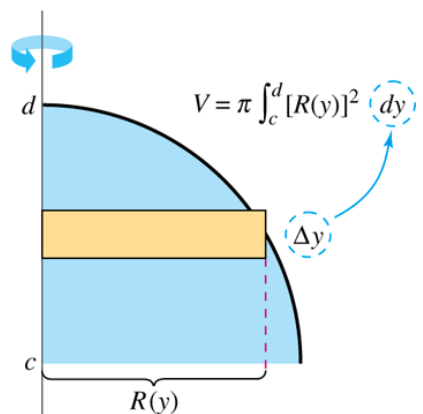
$$\text{Volume} = V = \pi \int_a^b [R(x)]^2 dx$$

Vertical Axis of Revolution

$$\text{Volume} = V = \pi \int_c^d [R(y)]^2 dy$$



Horizontal axis of revolution



Vertical axis of revolution

Definition of Arc Length

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve given by $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Definition of the Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2} dx \quad \text{y is a function of x.}$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2} dy \quad \text{x is a function of y.}$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.